An Improved Radiative Transfer Algorithm for Optically Thin Media

Colorado State University

CIRA
Cooperative Institute for Research in the Atmosphere

Brian McNoldy, Matthew Hopsecger,
Philip Gabriel, and Thomas Vonder Haar

Department of Atmospheric Science
Colorado State University

BACKGROUND

This study grew from a project in a *Radiative Transfer* course instructed by Philip Gabriel and Graeme Stephens at CSU in Fall 1999. The project involved Gaussian Quadrature, Phase Functions, and local bidirectional reflection and transmission functions.

From there, it was expanded to include global bidirectional reflection and transmission matrices using the "***Eigenmatrix Approach***", which takes advantage of the relationships between eigenvalues, eigenvectors, and the inverses of the eigenvectors.

It was at this point that Philip Gabriel suggested applying a power series expansion to a form of the Interaction Principle. After deriving that, an elegant new form of the global bidirectional matrices was found. However, by the nature of a power series, the "***Series Approach***" would be limited to small optical depths… although *how* small was unknown at the time.

Drawbacks of this method are that it is only an approximation (truncated infinite sum) and that it requires accurate knowledge of the optical depth τ (exponents on τ amplify any uncertainties). Advantages to this method are its symmetry (allowing for recursion) and the low number of terms required for high precision (fewer matrix operations).

The formulation of both methods will be presented in an abbreviated form here ("***Doubling***" is used in the comparison, but will not be shown in the Formulation section in the interest of space). The utility of this study is to arrive at the most streamlined radiative transfer algorithm… in this case, a retrieval would automatically select a particular method, based on optical depth, with a goal of shaving seconds or even minutes off of near-real-time calculations.

CONTACT:

mcnoldy@cira.colostate.edu

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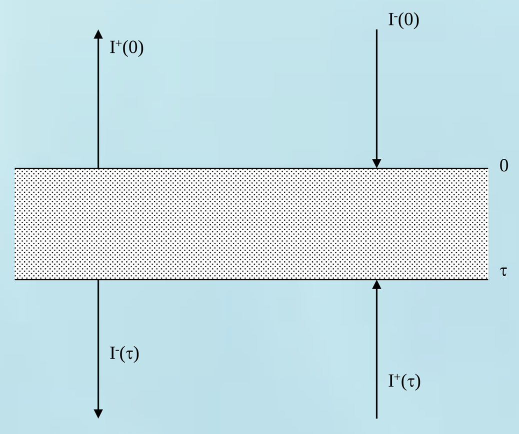
FORMULATION



For the Eigenmatrix and Series Approaches, a basic form of the Interaction Principle will be used, namely:

$I_{OUT} = f\,(I_{IN})$ such that $I(\tau) = e^{A\tau} I(0)$

In other words, the radiation leaving a layer of atmosphere can be expressed as a function of the radiation entering the atmosphere, given that the radiation incident at the bottom of the layer is a "propagation function" of the radiation incident at the top of the layer.

Eigenmatrix Approach:

Let $e^{A\tau} = e^{(XkX^{-1})\tau}$, since for any vector A, eigenvector X, and eigenvalue(s) k, $AX = Xk \Rightarrow A = XkX^{-1}$. Then $e^{A\tau} = e^{(XkX^{-1})\tau}$.

Using a series expansion, $e^{(XkX^{-1})\tau} \approx 1 + (XkX^{-1})\tau + (XkX^{-1})(XkX^{-1})\frac{\tau^2}{2!} + \ldots \approx 1 + (XkX^{-1})\tau + (Xk^2X^{-1})\frac{\tau^2}{2!} + \ldots$

And then a series "compression": $e^{A\tau} = Xe^{k\tau}X^{-1}$, where $X = \begin{pmatrix} e(+) & e(-) \\ e(-) & e(+) \end{pmatrix}$ and $X^{-1} = \begin{pmatrix} f(+) & f(-) \\ f(-) & f(+) \end{pmatrix}$.

Then $e^{A\tau} = \begin{pmatrix} e(+) & e(-) \\ e(-) & e(+) \end{pmatrix}\begin{pmatrix} e^{k\tau} & 0 \\ 0 & e^{-k\tau} \end{pmatrix}\begin{pmatrix} f(+) & f(-) \\ f(-) & f(+) \end{pmatrix} = \begin{pmatrix} e(+)e^{k\tau}f(+) + e(-)e^{-k\tau}f(-) & e(+)e^{k\tau}f(-) + e(-)e^{-k\tau}f(+) \\ e(-)e^{k\tau}f(+) + e(+)e^{-k\tau}f(-) & e(-)e^{k\tau}f(-) + e(+)e^{-k\tau}f(+) \end{pmatrix} \equiv \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix}$

Since $I(\tau)=f(I(0))$, $\begin{pmatrix} I^+(\tau) \\ I^-(\tau) \end{pmatrix} = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix}\begin{pmatrix} I^+(0) \\ I^-(0) \end{pmatrix}$. Rearranging yields $\begin{pmatrix} I^-(\tau) \\ I^+(0) \end{pmatrix} = \begin{pmatrix} M_{-+}M_{++}^{-1} & M_{--} - M_{-+}M_{++}^{-1}M_{+-} \\ M_{++}^{-1} & -M_{++}^{-1}M_{+-} \end{pmatrix}\begin{pmatrix} I^+(\tau) \\ I^-(0) \end{pmatrix}$

Introducing the global bidirectional reflection/transmission matrix… $\begin{pmatrix} I^-(\tau) \\ I^+(0) \end{pmatrix} \equiv \begin{pmatrix} R(\tau,0) & T(0,\tau) \\ T(\tau,0) & R(0,\tau) \end{pmatrix}\begin{pmatrix} I^+(\tau) \\ I^-(0) \end{pmatrix}$

$$R = \left[e(-)e^{k\tau}f(+) + e(+)e^{-k\tau}f(-)\right]\left[e(+)e^{k\tau}f(+) + e(-)e^{-k\tau}f(-)\right]^{-1}$$

$$T = \left[e(+)e^{k\tau}f(+) + e(-)e^{-k\tau}f(-)\right]^{-1}$$

Series Approach:

Using the same form of the Interaction Principle as before, $\begin{pmatrix} I^-(\tau) \\ I^+(0) \end{pmatrix} \equiv \begin{pmatrix} R(\tau,0) & T(0,\tau) \\ T(\tau,0) & R(0,\tau) \end{pmatrix}\begin{pmatrix} I^+(\tau) \\ I^-(0) \end{pmatrix}$, and solving for $I(\tau)$:

$\begin{pmatrix} I^+(\tau) \\ I^-(\tau) \end{pmatrix} = \begin{pmatrix} T^{-1}(\tau,0) & -T^{-1}(\tau,0)R(0,\tau) \\ R(\tau,0)T^{-1}(\tau,0) & T(0,\tau) - R(\tau,0)T^{-1}(\tau,0)R(0,\tau) \end{pmatrix}\begin{pmatrix} I^+(0) \\ I^-(0) \end{pmatrix}$. Then $e^{A\tau} = \begin{pmatrix} T^{-1} & -T^{-1}R \\ RT^{-1} & T - RT^{-1}R \end{pmatrix}$ for a vert homogeneous layer.

Since $A = \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}$, $e^{A\tau} = e^{\begin{pmatrix} t & -r \\ r & -t \end{pmatrix}\tau} \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}\tau + \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}^2\frac{\tau^2}{2!} + \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}^3\frac{\tau^3}{3!} + \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}^4\frac{\tau^4}{4!} + \ldots$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}\tau + \begin{pmatrix} t^2-r^2 & rt-tr \\ rt-tr & t^2-r^2 \end{pmatrix}\frac{\tau^2}{2!} + \begin{pmatrix} t(t^2-r^2)-r(rt-tr) & t(rt-tr)-r(t^2-r^2) \\ r(t^2-r^2)-t(rt-tr) & r(rt-tr)-t(t^2-r^2) \end{pmatrix}\frac{\tau^3}{3!} + \begin{pmatrix} t(t^3-tr^2-r^2t+rtr)-r(rt^2-r^3-trt+t^2r) & t(trt-t^2r-rt^2+r^3)-r(r^2t-rtr-t^3+tr^2) \\ r(t^3-tr^2-r^2t+rtr)-t(rt^2-r^3-trt+t^2r) & r(trt-t^2r-rt^2+r^3)-t(r^2t-rtr-t^3+tr^2) \end{pmatrix}\frac{\tau^4}{4!} + \ldots$$

Define $\eta = r\tau - (rt-tr)\frac{\tau^2}{2!} - (t(rt-tr)-r(t^2-r^2))\frac{\tau^3}{3!} - (t(trt-t^2r-rt^2+r^3)-r(r^2t-rtr-t^3+tr^2))\frac{\tau^4}{4!} + \ldots$. Then $-T^{-1}R = -\eta \Rightarrow R = T\eta$.

From the previous equation for $e^{A\tau}$, $T^{-1} \cong 1 + t\tau + (t^2-r^2)\frac{\tau^2}{2!} - \left[t(t^2-r^2)-r(rt-tr)\right]\frac{\tau^3}{3!} + \left\{t\left[t(t^2-r^2)-r(rt-tr)\right] - r\left[r(t^2-r^2)-t(rt-tr)\right]\right\}\frac{\tau^4}{4!}$

$$T = 1 - t\tau + (t^2-r^2)\frac{\tau^2}{2!} - \left[t(t^2-r^2)-r(rt-tr)-tr^2-r^2t\right]\frac{\tau^3}{3!} + \left\{27t\left[t(t^2-r^2)-r(rt-tr)\right] + r\left[r(t^2-r^2)-t(rt-tr)\right] - 6(t^2-r^2)^2 - 24t^2r^2 - 24tr^2t\right\}\frac{\tau^4}{4!}$$

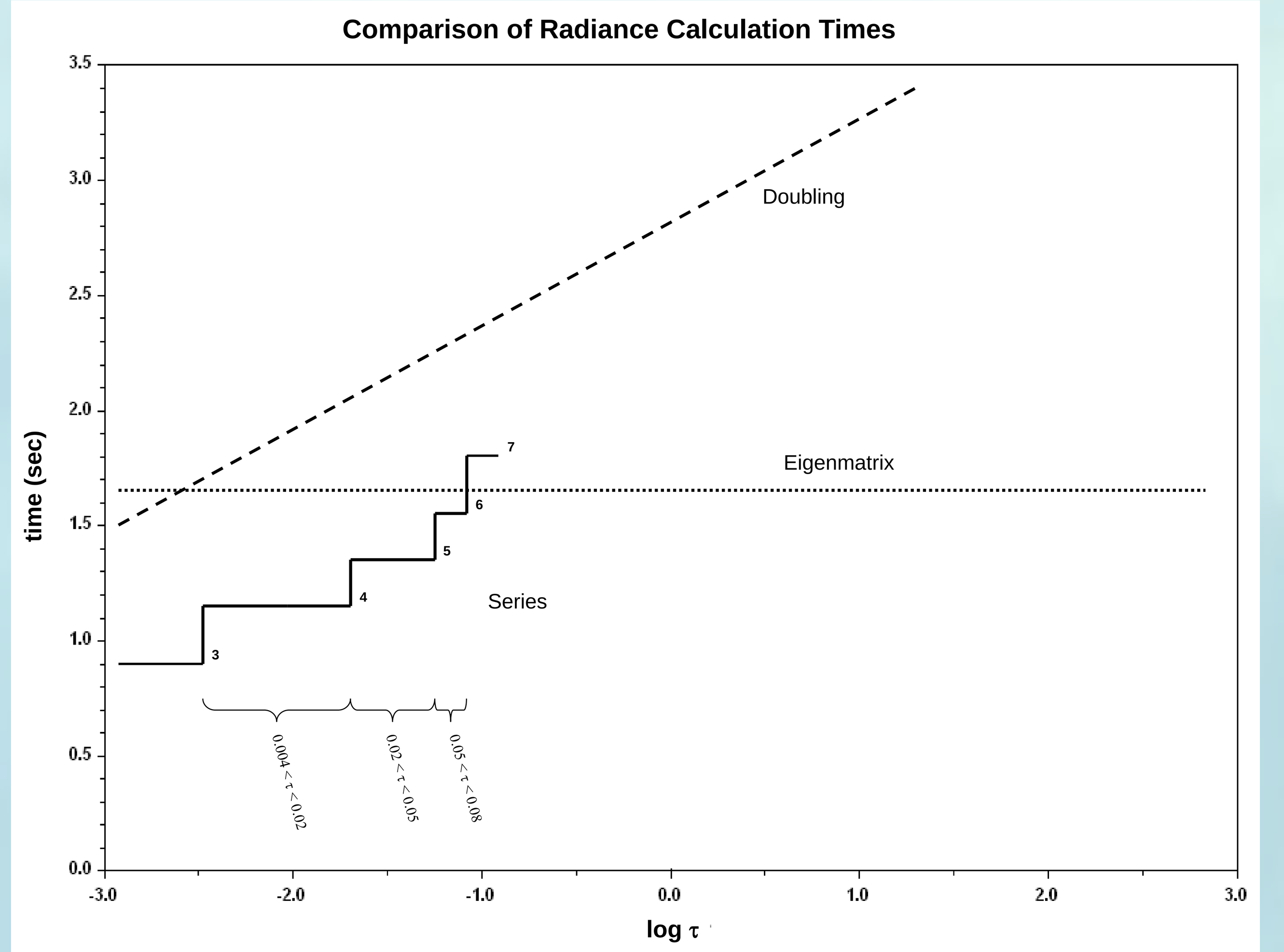
And recall $R = T\eta$, so

$$R = r\tau - (rt+tr)\frac{\tau^2}{2!} + \left[r(t^2-r^2)-t(rt-tr)+3(trt-t^3)\right]\frac{\tau^3}{3!} - \left\{t\left[r(t^2-r^2)-t(rt-tr)+2rt^2+6r^3+2trt\right] + r\left[t(t^2-r^2)-r(rt-tr)+4tr^2-6r^2t+16rtr\right]\right\}\frac{\tau^4}{4!}$$

RESULTS

The Eigenmatrix, Series, and Doubling Approaches were evaluated by computing radiances from each (500 times) and comparing the runtime. Doubling was always the slowest method for all but the smallest of optical depths. The Eigenmatrix Approach with the constant number of operations (matrix multiplications, inversions, etc) required the same time to run at all optical depths. The Series Approach, however, ran noticeably faster at $\tau < 0.1$.

Below is a graph showing how the methods compared as a function of optical depth. The times shown are worst-case; that is, each method was run for a wide range of asymmetry parameter (g) and single scatter albedo (ω_0) and the slowest time was used to create the graph.



The Series Approach can decrease computational time by as much as a factor of two, but even worst-case scenarios (shown above) yield a decrease of 50%.

How significant is this? Consider a retrieval with 30 layers and 1,000 wavenumbers. That requires 30,000 pairs of R and T matrices to be calculated. For example, if the optical depth is small enough, the retrieval runtime could be reduced from 3.4 minutes to only 2.5 minutes (maybe even 1.7 minutes if some general assumptions about ω_0 and g can be made).

The new Series Approach has a likely future on the CloudSat satellite (2003) to optimize onboard processing time for stratospheric and upper tropospheric retrievals.